1/12/26

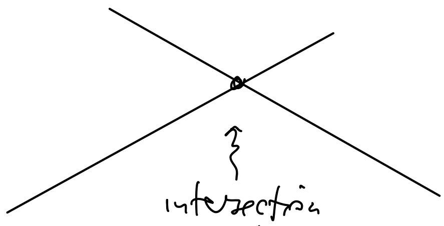
handout

Start with §1.1: Intro to Linear Systems

Motivation — Recall the equation of a line in the (x,y)-plane can be written in the form $ax+by=c$ (e.g. $2x+3y=4$). This is a <u>linear eqn</u> in 2 variables.

In high school or in calc. you learned how to find the <u>intersection</u> pt of 2 lines in $\mathbb{R}^2$:

line 1: $ax+by=c$
line 2: $dx+ey=f$



intersection pt.

In more variables it's harder to see the geometry behind this, so let's start with that.

Idea: manipulate the equations to solve for x, y.

<u>Ex</u> $\left|\begin{array}{l} 2x+3y=8 \\ 3x+y=5 \end{array}\right|$

Notice the notation

Manipulate the equations to get $x=1$, $y=2$

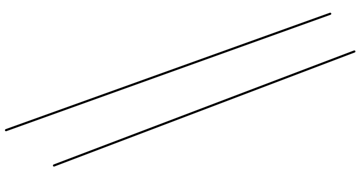
$\boxed{\text{We'll see a systematic way to do this shortly.}}$

Here's one solution: $E1 \times 3$, $E2 \times 2$

$$\begin{array}{r} 6x+9y=24 \\ -\,(6x+2y=10) \\ \hline 7y=14 \end{array} \qquad y=2,\ x=1$$

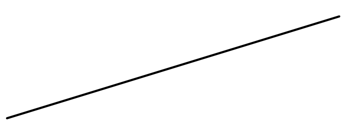
So usually 2 equations give a unique solution.

<u>Extreme situations</u> (exceptions):



- parallel lines — no sol'

e.g. $\left|\begin{array}{l} x+2y=4 \\ x+2y=5 \end{array}\right|$



- same line — infinitely many sol'ns

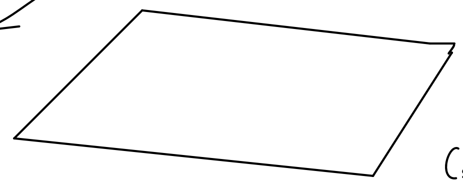
e.g. $2x+3y=4$
$4x+6y=8$

what if you're in $\mathbb{R}^3$? Now 3 variables

this is a linear equation in 3 vars

$\boxed{\text{one equation}}$ $\quad a_1x + b_1y + c_1z = d_1$

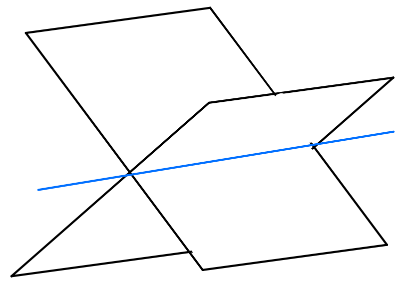
e.g. $2x + 3y + 4z = 5$



solution set = one plane

(∞ly many sol'ns)

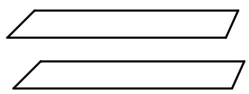
$\boxed{\text{2 equations}}$ $\quad \begin{vmatrix} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{vmatrix}$



(∞ly many sol'ns)

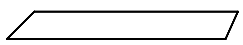
intersection = line (usually !)

Extreme situations (exceptions):



- parallel planes: no sol'n

e.g. $\begin{vmatrix} x + 2y + 3z = 4 \\ x + 2y + 3z = 5 \end{vmatrix}$



- same plane – the solution set is "2-dim'l"

e.g. $\begin{vmatrix} x + 2y + 3z = 4 \\ 2x + 4y + 6z = 8 \end{vmatrix}$ $\quad$ (∞ly many sol'ns)

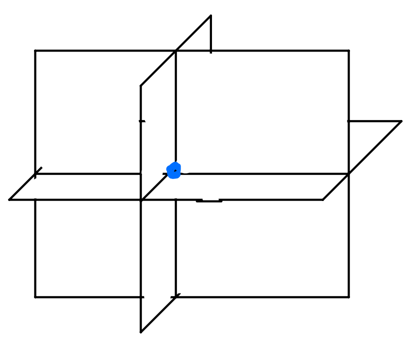
Issues (for 2 equations):

- Is there some uniform way to describe the line when that's the solution set?
- How do you even find that line ?

$\boxed{\text{3 equations}}$

general situation

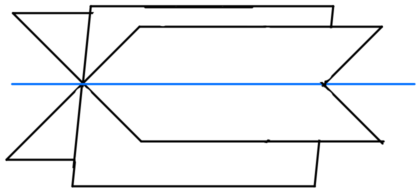
solution = one point !



(unique solution)

Extreme situations (exceptions):

Instead of the solution being a point, it might be

- a line  (∞ly many sol'ns)

- a plane (3 copies of the same plane — (∞ly many sol'ns)

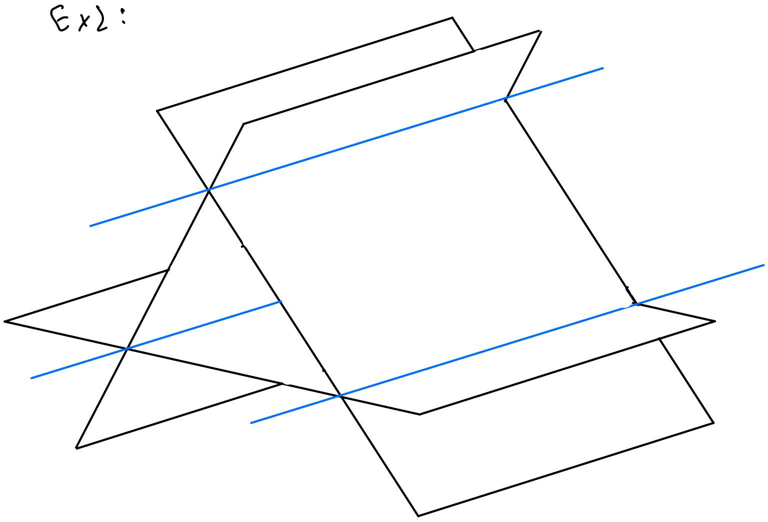
$$\text{e.g.} \quad \left| \begin{aligned} x+2y+3z&=4 \\ 2x+4y+6z&=8 \\ 3x+6y+9z&=12 \end{aligned} \right|$$

- empty

Ex 1: at least 2 of the planes are parallel

$$\text{e.g.} \quad \left. \left| \begin{aligned} x+y+z&=2 \\ x+y+z&=3 \\ 2x+3y+4z&=5 \end{aligned} \right| \leftarrow \text{irrelevant!} \right\} \text{ no solution}$$

Ex 2:



Issues

1. what happens when you go to more variables? Can't visualize.
   Switch to an algebraic approach.

2. Is there a uniform way to express the sol'n set?

3. Is there a uniform way to find the sol'n set?

Def a collection of linear equations is called a linear system.
Finding (all) the values of the variables that simultaneously satisfy all of the equations is called solving the linear system.

We'll talk about a systematic way to solve a linear system called
Gauss-Jordan elimination.

Ex (from before)

$$\left|\begin{array}{l} 2x + 3y = 8 \\ 3x + y = 5 \end{array}\right|$$

divide E1 by 2

$$\left|\begin{array}{l} x + \frac{3}{2}y = 4 \\ 3x + y = 5 \end{array}\right|$$

replace E2 by
$-3E1 + E2$

$$\left|\begin{array}{l} x + \frac{3}{2}y = 4 \\ 0x + \left(y - \frac{9}{2}y\right) = 5 - 12 \end{array}\right|$$

i.e.

$$\left|\begin{array}{l} x + \frac{3}{2}y = 4 \\ 0x - \frac{7}{2}y = -7 \end{array}\right|$$

mult E2 by $\frac{-2}{7}$

$$\left|\begin{array}{l} x + \frac{3}{2}y = 4 \\ 0x + y = 2 \end{array}\right|$$

replace E1 by

E1 $- \frac{3}{2}$ E2

$$\left|\begin{array}{l} x + 0y = 1 \\ 0x + y = 2 \end{array}\right|$$

$x = 1$

$y = 2$

Next insight: the variables are just place-holders. What matters is the numbers! Use matrices!!

$$\left[\begin{array}{cc|c} 2 & 3 & 8 \\ 3 & 1 & 5 \end{array}\right] \xrightarrow{\frac{1}{2}E1} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 4 \\ 3 & 1 & 5 \end{array}\right]$$

$$\xrightarrow{E2 \to -3E1 + E2} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 4 \\ 0 & -\frac{7}{2} & -7 \end{array}\right]$$

$$\xrightarrow{-\frac{2}{7}E2} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 4 \\ 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{E1 \to -\frac{3}{2}E2 + E1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

Goal is to get that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then we have $x = 1$
$y = 2$

Next time: Start with basic facts about matrices (incl. defs)